Reliability-Based Optimum Inspection and Maintenance Procedures

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The development of reliability-based optimum inspection and maintenance schedules for engines needs an understanding of the fatigue behavior of the engines. Critical areas of the engine structure prone to fatigue damage are usually identified beforehand or after the fleet has been put into operation. In these areas, fatigue cracks initiate after several flight hours, and these cracks grow in length until failure takes place when these cracks attain the critical lengths. Crack initiation time and its growth rate are considered to be random variables. Usually, the inspection (fatigue) or test data from similar engines are used as prior distributions. The existing state-of-the-art is to ignore the different lengths of cracks observed at various inspections and to consider only the fact that a crack existed (or did not exist) at the time of inspection. In this paper, a procedure has been developed to obtain the probability of finding a crack of a given size at a certain time if the probability distributions for crack initiation and rates of growth are known. Application of the developed stochastic models to devise optimum procedures for inspection and maintenance are also discussed.

Introduction

FATIGUE damage is one of the causes responsible for the deterioration of the reliability of the engine structure with use. This fatigue damage can be detected and corrected by selected time inspections and maintenance of the engines. The time between such inspections depends on the fatigue behavior of the engine and the desired reliability standards. More frequent inspections assure higher reliability standards but result in increased cost due to down time and inspection expenses. Cost benefits can be realized by optimizing the time between inspections. However, any methodology for the development of such optimization procedures needs an understanding and quantitative representation of the fatigue behavior of engines.

The present state-of-the-art does not provide any reliable techniques for estimating the fatigue behavior of the full-scale engine structures at the design state. Usually the fatigue behavior of the engine is estimated from the available inspection data on the particular engine, similar engines or test results. Based on the existing state-of-the-art, development of models for fatigue behavior from the inspection data is more reliable than and superior to the models developed from probabilistic load distribution, the consequent crack initiation and propagation. This is because at present, probability distribution of the load, fatigue crack growth under multiaxial stresses, and the inclusion of manufacturing defects in a fatigue model are not understood clearly.

Inspection data are obtained by checking for fatigue cracks at critical regions of the engine that are specially prone to fatigue damage. These regions can be identified before or after the fleet has been put into operation. Such critical areas are called "location stations" in this paper. Typical inspection data contain the identification numbers of the aircraft or engine structure, identification numbers of the location stations, the number of flight hours at the inspection time, length, and orientation of the observed cracks. Fatigue crack lengths at the same location station of a given fleet vary from engine to engine and exhibit a random behavior.

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Deterministic models are not in general suitable to analyze the inspection data and to develop quantitative models for the fatigue behavior from the data. Many attempts[1-17] have been made in the past to develop probabilistic models to describe fatigue failure qualitatively and quantitatively. Most of the investigations, however, have been restricted to the results of coupon tests. Some works, including Ref. 14, are concerned with the analysis of data from full scale aircraft wings.

In the usual development of probabilistic models, failure time has been defined as the number of the flight hours corresponding to that inspection time at which at least one crack, regardless of its size, is observed at a location station. Variations in the lengths of observed cracks are not included in the analysis. These variations can be attributed to the random character of fatigue crack initiation time and the differing flight hours at the time of inspection.

This paper describes the development of a probabilistic model for fatigue behavior that can incorporate the information on varying sizes of cracks at selected inspection times (at a given location station) on the engine. The model describes the complete stochastic process and includes the probability density functions of crack initiation and growth. Use of the model in quantitatively describing the observed data is also discussed in this paper. A methodology for devising an optimum scheme for inspection and maintenance is discussed on the basis of the developed stochastic model for fatigue behavior of the engine structure.

It is usually accepted that time (or cycles) to crack initiation is a random variable. After the crack initiation, the crack growth takes place only when the stresses favorable to such growth are present in the vicinity of the crack. Because of the nature of the use and operation of aircraft, such stresses, and hence the crack growth, exhibit a random behavior.

Integral Equation Model

A single critical location station is considered on the structure to develop this model. It is assumed that cracks can grow in units of ΔL. At any given instant of time, the crack length can be $r\Delta L$ where r is an integer. This assumption can be removed for this model by allowing r to tend to infinity. The time required for the development of a crack of length $r\Delta L$ is denoted by ω_r. The quantity T_r is the time needed for crack growth from its length $r\Delta L$ to $(r+1)\Delta L$. The time for the presence of a crack of length $(r+1)\Delta L$ is then

$$\omega_{r+1} = \omega_r + T_r \tag{1}$$

ω_r and T_r are random variables. The distribution for ω_r is assumed to be $F_r(t)$ and the density function for T_r is defined as $g_r(t)$. This distribution for time required for growth is assumed to be the same for all $r > 1$. The quantities T_r and the probability density function $g_r(t)$ for T_r can be related to the probability distribution of the random loads acting on the structure by fracture mechanics relationships. For example, the crack growth equation for 7075–76 aluminum alloy[18] is

$$\frac{da}{dn_c} = \frac{5 \times 10^{-13}[\sigma(\pi)a^{1/2}]^3}{2(6.8 \times 10^4) - \sigma(\pi)a^{1/2}} \tag{2}$$

at constant uniaxial stress level of σ which can be related to the random loads on the structure. However, at present it is not possible to include reasonably accurately the effects of multiaxial stresses and the variation in the stress levels. Furthermore, most of the results from fracture mechanics are valid for coupon tests while the real problem here is to develop fatigue models for full scale structures. Another difficulty in relating the distribution for T_r to stresses is in predicting accurately the probabilistic model for stress variations with time at the location station.

Because of these reasons, this paper describes the development of probabilistic models for fatigue behavior from the actual field inspection data or test data. Different distributions for T_r (that include crack initiation at $r = 1$) may exist. The methodology to be discussed in this paper is to estimate the parameters of hypothesized T_r's from the observed data. Based on these developments the effect of accuracy of specifying crack growth relations under multiaxial stresses, manufacturing defects, variation of loads in a full-scale engine structure can be studied.

Returning to the model development, the probability for the presence of a crack of length $(r+1)\Delta L$ is[19]

$$F_{r+1}(t) = P_r[\omega_{r+1} \leq t] = P_r[\mathrm{Max}(\omega_r + T_r), 0 \leq t] \tag{3}$$

or

$$F_{r+1}(t) = \int_0^t P_r[\omega_r + T_r \leq t | T_r = \tau]g_r(\tau)d\tau. \tag{4}$$

This expression can also be written as[19]

$$F_{r+1}(t) = \int_{\tau \leq t} F_r(t-\tau)g_r(\tau)d\tau. \tag{5}$$

It is further assumed that

$$F_1(t) = F_c(t) \quad t \geq 0 \tag{6}$$
$$= 0 \quad t < 0$$

where $F_c(t)$ is defined as the crack initiation probability. In other words, $F_c(t)$ denotes the probability distribution function for the time t for appearance of the first fatigue crack of length ΔL. Then,

$$F_2(t) = \int_{\tau \leq t} F_c(t-\tau)g_1(\tau)d\tau \tag{7}$$

If $f_c(t)$ is the density function for crack initiation the Eq. (7) can be rewritten as follows

$$F_c(t) = \int_{\tau \leq t}\left[\int_{\nu \leq t-\tau} f_c(\nu)d\nu\right]g_1(\tau)d\tau. \tag{8}$$

By introducing dummy variables u_1 and u_2, assuming f_c and g_1 are independent the following expression can be written for $F_2(t)$.

$$F_2(t) = \int_{u_2 \leq t,\, u_1+u_2 \leq t} f_c(u_1)g_1(u_2)du_1du_2 \tag{9}$$
$$= P[T_2 \leq t,\ T_1 + T_2 \leq t]$$

Similarly,

$$F_{r+1}(t) = P[T_1 \leq t,\ (T_1+T_2) \leq t, \ldots, (T_1 + T_2 + \ldots + T_r) \leq t] \tag{10}$$

A model such as this with arbitrary distributions for crack initiation and growth is of little practical use unless it is specialized for particular distributions. In the next section, a discrete growth model has been derived from basic principles by using a Poisson distribution for crack growth time and an arbitrary distribution for crack initiation. This development is similar to the simple stream equations of the Queueing Theory.[20,21] The applications of the derived model are discussed for exponential and Weibull distributions for crack initiation.

Poisson Growth Model

This model is first derived rigorously and later (following Eq. (20)) the derivation is explained by using simple physical reasoning. As before, a single critical location of the structure is considered, and it is assumed that cracks at this location can grow only in quantum lengths of magnitude ΔL. The value of ΔL can be assigned depending on the problem. The probability that at time t a crack of length $k(\Delta L)$ is observed is defined as $P(t;k)$. As a first step in the analysis, the probability that at time $(t + \Delta t)$ the length of crack is $k(\Delta L)$ is considered. This probability is denoted by $P[t + \Delta t;k]$. Poisson distribution for growth with parameters ν_k are assumed. The value of ν_k is the mean rate of crack growth from $k\Delta L$ to $(k+1)\Delta L$. For discrete values of k, $P(t + \Delta t;k)$ is possible in $(k+1)$ different ways. Then, from total probability theorem

$$P(t+\Delta t, k) = \sum_{j=0}^{k} P(t;j)\ P(\Delta t;k-j) \tag{11}$$

By introducing the notation

$$S_k = \sum_{j=0}^{k-2} P(t;j)\ P(\Delta t;k-j) \tag{12}$$

and by noting that $P(t;j)$ is never greater than one, the sum can be estimated

$$S_k \leq \sum_{j=0}^{k-2} P(t;j) = \sum_{m=2}^{k} P(\Delta t;m) \tag{13}$$

By letting $k \to \infty$

$$R_k = \sum_{m=2}^{\infty} P(m;\Delta t) = P(\Delta t;>1) \tag{14}$$

The quantity $P(>1, \Delta t)$ is the probability that the length of the crack increases by more than one ΔL during Δt. For orderly growth it is assumed that

$$[P(\Delta t;>1)/\Delta t] \longrightarrow 0 \text{ at } \Delta t \longrightarrow 0$$

Then, Eq. (11) can be written as

$$P(t+\Delta t;k) = P(k;t)\ P(\Delta t;0) + P(k-1;t)\ P(\Delta t;1) + P(\Delta t;>1) \tag{15}$$

It can be shown that for Poisson distribution

$$P(\Delta t;0) = 1 - \nu\Delta t + P(\Delta t;>1) \tag{16}$$

and

$$P(\Delta t;1) = \nu\Delta t + P(\Delta t;>1) \tag{17}$$

Then,

$$P(t + \Delta t;k) = P(k;t)(1 - \nu_k\Delta t) + P(k - 1;t)\nu_{k-1}\Delta t + 0[P(\Delta t, >1)] \quad (18)$$

The term 0[ ] denotes the order of the term in the parenthesis. By proceeding to limit as $\Delta t \to 0$, Eq. (18) becomes

$$P'(t;k) = \nu_{k-1}P[t;k - 1] - \nu_k P[t;k] \quad (19)$$

where

$$(dp[t;k]/dt) = P'[t;k] \quad (20)$$

The case $k = 1$ is a special case and is discussed later. Equation (19) can also be derived from simple physical reasoning as follows. At time $(t + \Delta t)$ a length of crack $k(\Delta L)$ can exist in the following two ways: a) The length of the crack is $(k - 1)\ \Delta L$ at time t, and the length of the crack increases by ΔL during Δt. Then the length of the crack at $(t + \Delta t)$ is $k(\Delta L)$. b) The length of the crack is $k(\Delta L)$ at time t, and the length of the crack does not increase during Δt. Then, again, the length of the crack is $k(\Delta L)$ at time $(t + \Delta t)$. Δt is assumed to be so short that these are the only possibilities and the two events are exhaustive and mutually exclusive. Then,

$$P[t + dt;k(\Delta L)] = P(A) + P(B) \quad (21)$$

$$P(A) = \nu_{k-1}\Delta t P[t;(k - 1)] \quad (22)$$

Similarly,

$$P(B) = (1 - \nu_k\Delta t)P[t;k] \quad (23)$$

and

$$P[t + \Delta t;k] = \nu_{k-1}\Delta t\, P[t;k - 1] + (1 - \nu_k\Delta t)\, P[t;k] \quad (24)$$

Dividing by Δt, proceeding to limit as Δt tends to zero, and using the notation Eq. (20) for $P'[t;k]$ Eq. (19) is again obtained.

The special case of $k = 1$ (length of the crack is ΔL) is that of crack initiation.‡ This case is possible in the following two ways: A_1—Crack does not exist at 't' and a crack appears during Δt; and B_1—A crack of length $1(\Delta L)$ exists at 't' and the crack does not grow during Δt. The probability $P(A_1)$ is the probability of crack initiation or time to first failure. Then,

$$P(t + \Delta t;1) = f_c(t)\Delta t + (1 - \nu_1\Delta t)\ P(t;k) \quad (25)$$

In Eq. (17), $f_c(t)$ is the probability density function for crack initiation. Then, for $k = 1$

$$P'(t;1) + \nu_1 P(t, 1) = f_c(t) \quad (26)$$

for $k > 1$,

$$P'(t;k) + \nu_k P(t;k) = \nu_{k-1}P(t;k - 1) \quad (27)$$

Solution for Exponential Distribution for $F_c(t)$

To illustrate the usefulness of the derived differential equations and the model for fatigue crack growth, an exponential distribution is hypothesized for crack initiation density $f_c(t)$. Then, the density function is

$$f_c(t) = \frac{1}{\beta}e^{-t/\beta} \quad (28)$$

By substituting $\beta = 1/\bar{\nu}$ the equation can be rewritten as follows

$$f_c(t) = \bar{\nu}e^{-\bar{\nu}t} \quad (29)$$

‡Crack initiation time has been defined as that time at which a fatigue crack of length ΔL has developed at the critical location of the structure.

For $f_c(t)$ given by Eq. (28), the solution $P(t,k)$ can be obtained from Eqs. (26) and (27) for the following initial conditions.

$$P(t, k) = 0, t = 0,\ k \geq 1 \quad (30)$$

The solution can be written as follows

$$P(t, k) = e^{-\nu_k t}\int_0^t \nu_{k-1}e^{(\nu_k-\nu_{k-1})t'_k}\int_0^{t'_k} \nu_{k-2}e^{(\nu_{k-1}-\nu_{k-2})t'_{k-1}} \cdots \int_0^{t'_2} \nu_1 e^{(\nu_2-\nu_1)t'_1}\int_0^{t'_1} \bar{\nu}e^{(\nu_1-\bar{\nu})\tau} \quad (31)$$

$$dt'_k dt'_{k-1}\ldots dt'_1 d\tau$$

$$k \geq 1$$

The integrals in the equation can be explicitly evaluated. However, the integral notation is useful for concise presentation and computational convenience. For

$$\nu_k = \nu_{k-1} = \ldots = \nu_1 = \nu, \quad (32)$$

The explicit solution after integration are as follows

$$P(t;k) = \frac{\nu}{\bar{\nu} - \nu}\nu^{k-1}e^{-\nu t}\left[\frac{1 - e^{(\nu-\bar{\nu})t}}{(\nu - \bar{\nu})^{k-1}} + \sum_{i=1}^{k-1}\frac{(-1)^{k-1-i}t^i}{i!\,(\bar{\nu} - \nu)^{k-1-i}}\right],$$

$$k \geq 1,\ t \geq 0; \quad (33)$$

$$P(t;1) = \frac{\bar{\nu}}{\bar{\nu} - \nu}e^{-\nu t}[1 - e^{(\nu-\bar{\nu})t}],\ t \geq 0$$

Solution for Weibull Distribution

In this case, the Poisson growth parameters (ν_i) are assumed not to vary with the crack length. The crack initiation distribution is assumed to be a two-parameter Weibull distribution. Then,

$$f_c(t) = \frac{\alpha}{\beta^\alpha}t^{\alpha-1}e^{-(t/\beta)^\alpha} \quad (34)$$

where α and β are the shape and scale parameters for the Weibull distribution. The Eqs. (26) and (27) can be solved for $P(t,k)$ for initial conditions $P(0,k) = 0$. The solution is

$$P(t;k) = \nu^{k-1}e^{-\nu t}\frac{\alpha}{\beta^\alpha}\int_0^t\int_0^{\tau'_{k-1}} \cdots \int_0^{\tau'_1} \tau^{\alpha-1}e^{\nu t-(\tau/\beta)^\alpha} \quad (35)$$

$$d\tau\ldots d\tau_{k-1};k \geq 1$$

These integrals, however, cannot be explicitly evaluated. The solution in the form of Eq. (35) can be used in the needed applications. One such application is the estimation for parameters of f_c(t) from the observed inspection data on cracks. This application is discussed and illustrated in the following sections.

Normalization

Before discussing the parameter estimation, normalization of $P(t,k)$ is considered in this section. In reality, the observed crack lengths do not always extend to infinity. Also, crack arresters are usually provided in many structures. This means that the probability $P(t,k)$ needs to be normalized by a realistic maximum length $N(\Delta L)$. If a symbol Z is used to denote

$$Z = \sum_{k=0}^{N} P(t;k) \quad (36)$$

the normalized $P(t,k)$ can be written as follows

$$P(t;k) = \frac{1}{Z}P(t;k) \quad (37)$$

Estimation of the Parameters of $f_c(t)$ (Exponential Distribution)

In this section a method is described for estimating the parameters of a hypothesized model for crack initiation by using the inspection data. For a single location station, the observed inspection data contain the identification number of the structure, flight hours at the inspection time, and the lengths of the observed cracks. Table 1 illustrates the data for the center box wing of a fleet of aircraft. The center box wing data are used because engine data are not available to the authors at this time. Table 1 contains data from a small sample of 57 from the fleet size 104. For calculations discussed in this paper data from all the 104 aircraft are used. A typical column in this table can be interpreted as follows. At the location station ℓ a crack of length $k_i(\Delta L)$ is observed at time t_i. The probability $\rho_{\ell 1}$ of the occurrence of this event can be written as

$$\rho_{l1} = P[t = t_i; k = k_i] \tag{38}$$

Probability $P_{\ell 1}$ that a crack of length $k_i(\Delta L)$ is observed for $t \leq t_i$ can be obtained from Eqs. (31), (35) or (36). For a hypothesized exponential distribution for $f_c(t)$, growth probability density parameter $\nu = \nu_k$ and for a crack of length $k(\Delta L)$, $P_{\ell 1}$ can be written as follows

$$P_{l1} = P[l \leq t_i; k = k_i] = \left\{ e^{-\nu_k t_i} \int_0^{t_i} \nu_{k-1} e^{(\nu_k - \nu_{k-1})t'_k} \times \int_0^{t'_{k-1}} \nu_{k-2} e^{(\nu_{k-1}-\nu)t'_{k-2}} \int_0^{t'_2} \nu_1 e^{(\nu_2 - \nu)t_1} \int_0^{t_1} \bar{\nu} e^{(\nu_1 - \bar{\nu})\tau} \times dt'_k dt'_k \ldots dt'_1 d\tau \right\} \frac{1}{(Z)_{t=t_i}} \tag{39}$$

From this equation $\rho_{\ell 1}$ can be obtained

$$\rho_{l1} = P[t = t_i; k = k_i] = P'(t_i, k_i) = \frac{d}{dt} P(t, k_i) dt] t = t_i \tag{40}$$

For 'n' independent observations, the likelihood function[22] can be written as follows,

$$L[\bar{\nu} | t_1, t_2 \ldots t_j \ldots t_n \ldots ; k_1 k_2 \ldots k_j, k_n] = \prod_{j=1}^{n} P'(t_j, k_j) \tag{41}$$

Then,

$$ln(L) = \sum_{j=1}^{n} ln P'(t_j, k_j) \tag{42}$$

The maximum likelihood equation for estimation of the parameter $\bar{\nu}$ can be obtained by equating the first derivative of $\ell n(L)$ with respect to $\bar{\nu}$ to zero, i.e.,

$$\frac{\partial}{\partial \bar{\nu}} (l n L) = 0 \tag{43}$$

In this case the equation is as follows

$$\sum_{j=1}^{n} \left\{ \frac{\frac{\partial}{\partial \bar{\nu}} [P'(t_j;k_j) - P(t_j;k_j) Z'/Z]}{P'(t_j;k_j) - P(t_j k_j) Z'/Z} - \frac{1}{Z}\frac{dZ}{d\bar{\nu}} = 0 \tag{44}$$

This equation can be solved to obtain $\bar{\nu}$. It can be shown in the usual way that this equation possesses only one positive root.[22] Without normalization, the equation can be simplified to the following form

$$\frac{1}{\bar{\nu}} = \beta = \frac{1}{n} \sum_{i=1}^{n} \frac{A_{k-1} - A_k}{B_{k-1} - B_k} \tag{45}$$

where

$$A_i = \nu_i e^{-\nu_i t} \int_0^t \eta_{i-1} \int_0^{\tau_{i-1}} \eta_{i-2} \ldots \int_0^{\tau_1} \tau_0 e^{(\nu_1 - \frac{1}{\beta})t_0} d\tau_0; \ i \geq 1 \tag{46}$$

$$= t e^{-t/\beta}; \ i = 0$$

$$B_i = \nu_i e^{-\nu_i t} \int_0^t \eta_{i-1} \int_0^{\tau_{i-1}} \eta_{i-2} \ldots \int_0^{\tau_1} e^{(\nu_1 - \frac{1}{\beta})\tau_0} d_{\tau_0}, \ i \geq 1 \tag{47}$$

$$= e^{-t/\beta}; \ i = 0$$

$$\eta_i = \nu_i e^{(\nu_{i+1} - \nu_i)\tau_i} d\tau_i \quad i \geq 1 \tag{48}$$

$$= \frac{1}{\beta} e^{(\nu_{i+1} - \frac{1}{\beta})} \tau_i d\tau_i \quad i = 0$$

This Eq. (44) or (45) can be solved for $\beta = (1/\bar{\nu})$ by the method of successive substitution.

Maximum Likelihood Method for Weibull $f_c(t)$

In this section the method of maximum likelihood[22] is described for a hypothesized two-parameter Weibull distribution (Eq. 34) for $f_c(t)$. The probability $\rho_{\ell 1}$ that a crack of length $k_i(\Delta L)$ is observed at $t = t_i$ is again given by Eq. (40). Then, from Eq. (35) to Eqs. (37) and (40)

$$\rho_{l1} = P[t = t; k = i] = \frac{\alpha}{\beta^\alpha} \{ \nu^{i-1} H(i-1) - \nu^i H(i) \} dt \ k \geq 1 \tag{49}$$

where

$$H_i = e^{-\nu t} \int_0^{t'} \int_0^{\tau_{i-1}} \int_0^{\tau_{i-2}} \ldots \int_0^{\tau_1} \tau_0^{\alpha-1} e^{\nu\tau_0} e^{-(\tau_0/\beta)^\alpha} \times d\tau_0 d\tau_1 \ldots d\tau \tag{50}$$

and

$$P[t \leq t_1; k = k_i] = \frac{\alpha}{\beta^\alpha} H_k \quad k \geq 1 \tag{51}$$

Then

$$L(\alpha, \beta | t_1, t_2 \ldots t_j \ldots t_n; k_1, k_2 \ldots k_j \ldots k_n) = \prod_{j=1}^{n} P'(t_j; k_j) \tag{52}$$

The quantity $\ell n(L)$ can be written as

$$lnL = \sum_{j=1}^{n} P'(t_j, k_j) \tag{53}$$

By using the necessary conditions that L is maximum the following equations can be obtained for α and β

$$n\left(\frac{1}{\alpha} - ln\beta\right) + \sum_{j=1}^{n} \left\{ \frac{H,_\alpha(k_j - 1) - Q,_\alpha(k_j)}{H(k_j - 1) - Q(k_j)} - \frac{1}{Z}\frac{\partial Z}{\partial \alpha} \right\} = 0 \tag{54}$$

$$-n\frac{\alpha}{\beta} + \sum_{j=1}^{n} \left\{ \frac{H,_\beta(k_j - 1) - Q_\beta(k_j)}{H(k_j - 1) - Q(k_j)} - \frac{1}{Z}\frac{\partial Z}{\partial \beta} \right\} = 0 \tag{55}$$

where

$$H(i) = \nu^i e^{-\nu t} \int_0^t \int_0^{\tau_{i-1}} \ldots \int_0^{\tau} \tau_0^{\alpha-1} e^{-\nu\tau_0} e^{-(\tau_0/\beta)^\alpha} d\tau_0 \ldots d\tau_{i-1}, \tag{56}$$

$$Q(i) = [H(i)][1 + Z'/Z\nu] \tag{57}$$

The quantities $H,_\alpha$, $H,_\beta$ are partial derivatives of H with respect to α and β, respectively. Similarly, $Q,_\alpha$ and $Q,_\beta$ are the appropriate partial derivatives of Q. Numerical solutions of these equations are illustrated in the following sections.

Estimation of Parameters of $f_c(t)$ and Growth Distribution Parameters ν_i

This procedure of estimating parameters $\bar{\nu}$ and $\bar{\nu}_i$'s is illustrated for exponential distribution for $f_c(t)$. The data in

Table 1 Fatigue inspection data at a critical region from a fleet

Serial no.	Time (1000 hr)	Crack length (in.)	k
1	4.246	0.12	3
2	4.904	0.31	8
3	4.972	0.62	15
4	4.349	0.06	1
5	5.072	0.12	3
6	4.888	0.03	1
7	4.653	0.25	6
8	5.014	0.09	2
9	4.484	0.12	3
10	4.574	0.12	3
11	4.722	0.03	1
12	4.268	0.03	1
13	4.914	0.12	3
14	4.350	0.12	3
15	3.774	0.09	2
16	4.705	0.03	1
17	4.405	0.03	1
18	4.944	0.03	1
19	3.663	0.06	1
20	4.415	0.06	1
21	4.595	0.06	1
22	4.064	0.06	1
23	3.951	0.12	3
24	5.152	0.12	3
25	3.506	0.06	1
26	4.400	0.12	3
27	4.624	0.06	1
28	4.731	0.03	1
29	5.620	0.12	3
30	5.374	0.12	3
31	4.292	0.06	1
32	4.091	0.06	1
33	7.332	0.06	1
34	4.505	0.12	3
35	4.574	0.12	3
36	4.623	0.12	3
37	4.100	0.06	1
38	4.893	0.09	2
39	4.148	0.19	5
40	4.719	0.09	2
41	4.432	0.03	1
42	4.358	0.03	1
43	4.710	0.03	1
44	4.630	0.06	1
45	4.691	0.25	6
46	4.239	0.12	3
47	4.215	0.25	6
48	4.516	0.06	1
49	4.478	0.12	3
50	5.110	0.03	1
51	3.757	0.06	1
52	4.516	0.03	1
53	4.396	0.06	1
54	4.682	0.06	1
55	4.627	0.15	4
56	3.877	0.06	1
57	3.858	0.06	1

Table 1 contain the times of observation t_i, values crack lengths ℓ_i, and the corresponding values of k_i. The probability p_{ℓ_1} of observing a crack of length $= k_i(\Delta L)$ at $t = t$ is given by Eq. (38). In the previous sections, the parameters ν_i's are assumed to be known. If these parameters are unknown the likelihood function for n observations can be written as follows. In this case the parameters of $f_c(t)$ and the growth distribution parameters ν_i's are unknown.

$$L_3(\bar{\nu}, \nu_1, \nu_2 \ldots \nu_n \mid t_1, t_2 \ldots t_j \ldots t_n; k_1 \ldots k_j \ldots k_n) = \prod_{j=1}^{n} P'(t_j;k_j) \tag{58}$$

Table 2 Fatigue crack data from a test

Serial no.	Time (10^4 cycles)	Crack length	k
1	1.6	0.061	2
2	2.5	0.092	3
3	2.2	0.081	3
4	2.7	0.120	4
5	1.8	0.068	2
6	4.8	0.220	8
7	0.5	0.027	1
8	0.8	0.028	1
9	2.8	0.110	4
10	3.4	0.130	4
11	3.6	0.190	7
12	3.9	0.150	7
13	1.3	0.029	1
14	1.2	0.035	1
15	0.2	0.029	1
16	5.8	0.270	10
17	3.1	0.150	5
18	1.5	0.075	2
19	8.0	0.330	12
20	3.2	0.130	4
21	3.7	0.140	5
22	0.6	0.061	2
23	4.2	0.190	7
24	2.9	0.180	6
25	7.2	0.280	10
26	0.3	0.027	1
27	3.7	0.150	5
28	2.6	0.074	2
29	1.2	0.035	1
30	2.4	0.100	3

Then,

$$lnL_3 = \sum_{i=1}^{n} lnP'(t_j;k_j) \tag{59}$$

The $(n + 1)$ unknowns can be determined from the following $(n + 1)$ equations

$$\begin{aligned} \frac{\partial}{\partial \nu}(lnL_3) &= 0 \\ \frac{\partial}{\partial \nu_1}(lnL_3) &= 0 \\ &\vdots \\ \frac{\partial}{\partial \nu_n}(lnL_3) &= 0 \end{aligned} \tag{60}$$

For accurate determination of the values of $\bar{\nu}$ and ν_i's sufficient observations are needed at various crack lengths.

Numerical Examples

The maximum likelihood Eq. (44) is solved for $\bar{\nu}$ for the data of Tables 1 and 2. Data from Table 1 are from inspections of the center box wing of a fleet of aircraft. These data are used because the data on engine structure are not available to the authors at the time. Data of Table 2 could have resulted from fatigue tests. The method of successive substitutions[23] is used to solve the Eq. (44). The integrals needed in the equation are evaluated by matrix multiplication technique.[24] Computer programs are written to carry out the necessary computations.

Case I

Data from Table 2 are first studied because of the simplicity and the small number of items. Crack lengths in

Table 3 Cumulative probabilities at given values of time

k	$P(0.2,k)$	$P(0.5,k)$
0	0.54542	0.219698
1	0.38684	0.500221
2	0.06098	0.211283
3	0.00625	0.056010
4	0.00048	0.010878
5	0.00003	0.001671
6	...	0.000212
7	...	0.000023
8	...	0.000003
9–19	...	...
$\sum_{k=0}^{20} P(t,k)$	1.0	1.0

whether a crack is observed at the time of inspection. If a crack is observed cycles to failure are noted. An exponential distribution is hypothesized for the cycles to failure, i.e.,

$$P(t) = 1 - e^{-\bar{\nu}_1 t} = 1 - e^{t/\beta_1}$$

The quantity β_1 is defined as the characteristic number of cycles to failure. The parameter $\bar{\nu}$ can be estimated by the method of maximum likelihood. The values are

$$\bar{\nu}_1 = 0.36(10^4 \text{ cycles})^{-1}$$

and

$$\beta_1 = 2.79(10^4 \text{ cycles})$$

Case II

Again, data from Table 2 are used, but varying lengths of cracks at the observed time are taken into account. As previously discussed, an exponential distribution $f_c(t)$ is hypothesized for crack initiation. The mean rate of crack growth is assumed to be 0.0435 in. per 10^4 cycles. The parameter ν for the growth distribution represents the mean rate of crack growth. If the parameters ν_i's are assumed not to vary with crack length the value of ν depends on the assumed value of ΔL. In this case ΔL is assumed to be equal to 0.027 in. Then

$$\nu = \frac{0.0435}{0.027}(10^4 \text{ cycles})^{-1} = 1.5\ (10^4 \text{ cycles})^{-1}$$

Column 4 in Table 2 shows the k values in the term $k(\Delta L)$ that are used in the development of the theory. The parameter β is estimated from Eq. (44). The normalizing number N is assumed to be 20. Then,

$$\bar{\nu} = 3.031\ (10^4 \text{ cycles})^{-1}$$

$$\beta = 0.32\ (10^4 \text{ cycles})$$

The characteristic number of cycles to get a crack size of 0.027 in. is 3200 cycles while the calculation for the same data in case I suggests that the characteristic number of cycles for crack appearance is 27,900 cycles. The large number of cycles in case I is because crack initiation is not considered. It is the characteristic number of cycles to failure where all the crack lengths for 0.027 in.–0.280 in., are considered to be lumped into one group to define failure. The analysis of case II by the method developed in this paper provides a characteristic time for initiation of cracks of size 0.027 in. In addition, for future inspection purposes, the characteristic number of cycles for appearance of a crack of length 0.135 in. or any other length that is needed, can be calculated from Eqs. (26) and (27). The analysis of case I does not provide this information. The only information from case I is that at the characteristic number of cycles (27,900) a crack of any length may appear.

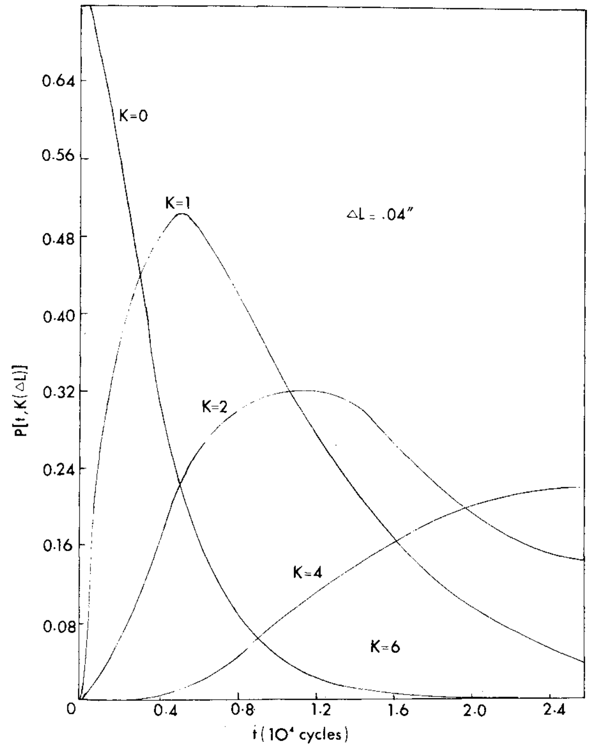

Fig. 1 Probability functions vs time.

It is also known that at any selected number of cycles 't' there can be no crack, a crack of length $1(\Delta L)$, a crack of length $2(\Delta L)$, . . . , a crack of length $N(\Delta L)$. No state, other than these, is possible. Then,

$$\sum_{k=0}^{N} P(t;k) = 1$$

This is used as a check on the calculations. Table 3 shows $P[0.2(10^4 \text{ cycles}),k]$ and $P[0.5(10^4 \text{ cycles}),k]$ and their sum for values of k = 0–20. These sums add to be equal to unity as expected. Fig. 1 shows a plot of $P(t;1)$, $P(t;2)$, $P(t;3)$, $P(t;4)$, $P(t;5)$, and $P(t;6)$ vs cycles.

Case III

The data from Table I are analyzed by assuming of ΔL to be 0.04 in. Calculations can be carried out for other values of ΔL including those that are much smaller than 0.04. The parameters ν_i are assumed to be known for different values of i. The parameters based on test results and assumed number of cycles of 300 per hour are shown in Table 4. The parameter β is calculated by solving Eq. (44) and assuming N to be 20. This value of β = 2261 hr. This is the characteristic number of flight hours for initiation of crack of size 0.04 in. If a crack of a length of 0.32 in. can be tolerated in a particular aircraft without the

Table 4 Poisson parameter in crack growth distribution function of the crack length

Crack length (in.)	ν_k (ΔL in./1000 hr)
0.04	1
0.08	2
0.12	2
0.16	3
0.20	4
0.24	5
0.28	6
0.32	7

danger of violating the reliability standards, the inspections can be scheduled when the probability for the number of hours for appearance of a crack of length 0.32 in. is known. This can be calculated from Eq. (27) when $\beta = 1/\bar{\nu}$ is known. This freedom is not available when varying lengths of cracks are ignored.

Case IV

The set of data in Table 1 is again analyzed by hypothesizing a 2-parameter Weibull distribution for crack initiation probability $f_c(t)$. Equations (54) and (55) which consider the variations of crack lengths but assumes all ν_i's are the same are solved for the parameters α and β. An average value of $\nu = 2.3$ (times ΔL in./10000 cycles) is used in the calculations. The calculated values are $\alpha = 9.8$ and $\beta = 2300$ hr. The value of α is much greater than 1 and this suggests that the exponential distribution is not a suitable distribution for crack initiation at least in the case when all ν_i's are assumed to be equal to ν.

The accuracy of representation of the particular set of data by the developed model can be established by appropriate significance tests. The development and application of these significance tests will be discussed elsewhere.

Optimum Schedules for Inspection and Maintenance

A methodology for developing optimum schemes for inspection and maintenance are discussed in this section. This methodology is based on the stochastic models developed for fatigue behavior in this paper. A single critical location station is considered for clarity in explanations. The desired design life of the engine structure is denoted by '*DL*' hours. It is assumed that the structure is inspected periodically at intervals of H number of hours. The frequency of inspection is denoted by 'I_n'

$$I_n = (DL/H) \tag{61}$$

At each inspection, the critical location station is checked for fatigue cracks. If a crack of length $k\Delta L > k^* \Delta L$ is observed the observed crack is repaired. The engine structure is assumed to be as good as new after repair. The particular length $k^* \Delta L$ is a quantity assumed to be predetermined. On the other hand, if a crack is not observed at the critical location or if the length of the observed crack is less than $k^* \Delta L$ no repairs are done.

The total cost model for optimization purposes can be written as follows

$$\text{Cost} = C_1 + C_2 \tag{62}$$

where C_1 is the cost due to inspection and maintenance. C_2 is the cost due to failure

$$C_1 = F_n I_n S_c \tag{63}$$

where F_n is the fleet size, I_n is the number of inspections and S_c is the cost of inspection and maintenance of a single structure.

$$C_2 = F_n (PI_n) C_f \tag{64}$$

where (PI_n) is the probability of failure under I_n inspections and C_f is the cost of failure of one structure. The probability (PI_n) can be calculated in the following way. Probability of failure in the first interval is denoted by P_1. Then,

$$P1 = P(t \leq H;\ k \geq k^c) \tag{65}$$

$$P1 = 1 - \{P(H;0) + P(H;1) \ldots + P(H;k^c - 1)\} \tag{66}$$

Failure is assumed to take place when $k\Delta L \geq k^c \Delta L$. The probability of failure P_2 in the second interval is given by the following two conditions. The length of the crack at the first inspection could be greater than or equal to $k^*\Delta L$. This crack is then repaired and the structure is as good as new. If the crack length is less than $k^*\Delta L$ nothing is done. Detection probability of a crack of length $k^*\Delta L$ is assumed to be unity. Then,

$$P2 = P[2H_R, K \geq K^c | H, K \geq K^*] + P[2H, K \geq K^c | H, K < K^*] \tag{67}$$

The first term in this equation represents the probability of failure in the 2nd interval given that a crack of length $k^*(\Delta L)$ is found at the first inspection at the location station and the structure is repaired. The second term represents the probability of failure in the 2nd interval given that no crack or a crack of length less than $k^*(\Delta L)$ is found at the first inspection. Similarily, the probability of failure at the I_nth inspection can be obtained as follows. This probability depends on whether a crack of length greater than $k^*\Delta L$ is never observed or at the first inspection, second inspection or . . . at the $(n\text{-}1)$th inspection. For example,

$$\begin{aligned} P3 = {} & P[3H_R, K \geq K^c | 2H, K \geq K^* | H, K \geq K^*] \\ & + P[3H_R, K \geq K^c | 2H, K \geq K^* | H, K < K^*] \\ & + P[3H, K \geq K^c | 2H, K < K^* | H, K < K^*] \\ & + P[3H_R, K \geq K^c | 2H, K < K^* | H, K \geq K^*] \end{aligned} \tag{68}$$

The probabilities $P1$ to PI_n need new expressions when the detection probabilities are included. The probability of detecting a crack of length $k^*\Delta L$ when a crack of that length is present at the location station is assumed to be F_D. This probability depends on the nondestructive inspection capabilities. For example, $P2$ can be written as follows if the detection probabilities are included.

$$P2 = F_D P[2H_R, K \geq K^c | H, K \geq K^*] + (1 - F_D) P[2H, K \geq K^c | K < K^*] \tag{69}$$

In this equation, the first term represents the probability of failure given that the crack of length $k \geq k^*$ is detected and repaired during the first inspection. The second term represents the probability that the crack of length $k \geq k^*$ has been in existence during the first inspection but is not detected. The third term represents that no crack of length $k \geq k^*$ has been in existence at the first inspection. Similarly the expression for PI_n can be written.

Now, the cost function [Eq. (62)] can be minimized to obtain the optimum frequency I_n. There is, however, a restraint which is necessary. Reliability R based on the probability of first failure P_f in a fleet size F_n is restricted to be above an acceptable bound R_b. Depending on the severity of the problem and the bounds on reliability based on higher order failures can be considered. By usual methods[25] R_f can be expressed in terms of PI_n. Then,

$$R = 1 - P_f = [1 - PI_n] > R_b \tag{70}$$

where R_b is the bound on first failure probability. The minimization procedures are standard and are discussed in literature on operations research. There are three parameters of interest in this problem. The ratio of cost of inspection to cost of failure (S_c/C_f), the parameter k^* and the detection probability F_D. The parameter k^* and the amount of money and effort to be spent on inspection can also be controlled. Optimization procedures can be carried out with varying k^*, S_c/C_f and F_D to determine the best possible specifications and frequency of inspection. Trade off between improving nondestructive inspection capability (F_D) and k^* is possible.

Conclusions

This paper describes the development of a model for the fatigue behavior of engine structures. This model includes the stochastic process of crack initiation and growth probabilities. The Poisson growth model discussed in detail in this paper has the freedom of selecting or hypothesizing a model for initiation of crack of size ΔL. The integral equation model has the additional freedom of choice of growth distributions.

The methodology for quantitative estimation of the parameters of the developed stochastic model from observed inspection data (times of observation and crack lengths) is discussed and illustrated by examples. Methods of checking the accuracy of the quantitative models and the needed significance tests are not reported in this paper. Application of the stochastic model to develop optimum plans for inspection and maintenance is discussed. The methodology developed in this paper is also applicable to types of structures other than the engine structures.

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